

# A Hamiltonian approach to implicit systems, generalized solutions and applications in optimization

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## Abstract

We introduce a constructive method that provides the local solution of general implicit systems in arbitrary dimension via Hamiltonian type equations. A variant of this approach constructs parametrizations of the manifold, extending the usual implicit functions solution. We also discuss the critical case of the implicit functions theorem, define the notion of *generalized solution* and prove existence and properties. Examples are also indicated. The applications concern necessary conditions and algorithms in nonconvex optimization problems and their perturbations.

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## 1 Introduction

In the Euclidean space  $R^d$ ,  $d \in N$ , we consider the general implicit functions system:

[illegible]

where  $l \in N$ ,  $1 \leq l \leq d-1$  and  $F_j \in C^1(\Omega)$ ,  $F_j(x^0) = 0$ ,  $j = \overline{1, l}$ ,  $x^0 \in \Omega \subset R^d$  bounded domain, given.

The problem (1) has a long and well known history and we quote the monographs of Krantz and Parks [14], Dontchev and Rockafellar [10] for a comprehensive presentation, including important applications and recent research developments. We also mention the book by Thorpe [28], where related ideas are discussed from the point of view of differential geometry. In particular, it is known that one can associate to (1) a system of nonlinear (partial) differential equations (basically derived from the differentiation formula, under usual assumptions), see [14], Ch. 4.1.

In the recent paper [29], in dimension two and three, it was shown that one can associate to (1) other (essentially simpler) systems of ordinary differential equations, under the mere assumption that  $F_j \in C^1(\Omega)$ ,  $j = \overline{1, l}$  and in the absence of any independence-type condition. These new systems provide a constructive (local) parametrization of the solution of (1) around  $x^0$  under certain conditions.

Moreover, it is possible to define a local generalized solution of (1) even in the critical case, in arbitrary dimension. Our approach to this old question is novel . A variant (in dimension three) was discussed in [19] as well, where it was proved that it is enough to use ordinary differential systems (especially of Hamiltonian type) in order to solve locally (1) via appropriate parametrizations of the unknowns. Some relevant numerical examples are also indicated.

In this paper, we discuss the solution of the general implicit system (1) in arbitrary dimension, by using a new iterated system of ordinary differential equations. The approach has a constructive character and we indicate two variants that give a parametrization of the unknowns in (1) or construct exactly the classical implicit solution (in function form, see Theorem 7). This is done in Section 2, under the usual nondegeneracy condition from

the implicit function theorem. We underline that the existence question is well known (via the classical implicit functions theorem or the inverse function theorem, etc.), but a general and effective construction seems not to be available, to the best of our knowledge. The systems of ordinary differential equations that we use here are derived from a first order partial differential system of equations. As a first application, in the final part of Section 2, perturbations of the system (1) are investigated, both for implicit functions and implicit parametrizations.

We also recall that, in algebraic geometry, implicitization and parametrization (via rational functions) are important questions, Gao [11], Wang [30], Schicho [25]. General parametrization methods are not known, Gao [11]. Recent papers study approximate parametrization approaches, Dobiasova [9], Yang, Jüttler and Gonzales-Vega [31].

In Section 3, we show how to solve the critical case as well, under the mere assumption  $F_j \in C^1(\Omega)$ ,  $j = \overline{1, l}$ . We introduce the notion of (local) *generalized solution*, prove its existence and properties. We also indicate some relevant examples. The generalized solution obtained by our method covers all possible cases and is an extension of the notion of local solution from the implicit function theorem, in the classical case. Singular situations in the implicit functions theorem were discussed by different methods in [3], [7], [16] and a comprehensive account can be found in [14], Ch.5.4, where it is specified that a complete solution of the critical case is not known.

The last section is devoted to applications in classical nonlinear programming. We use reduced gradients to obtain optimality conditions in the simpler Fermat form, involving no or fewer multipliers. We also introduce an algorithm that works in the critical case as well. Some illustrative numerical examples are also provided.

## 2 Implicit parametrizations

In this section, we discuss the system (1) under the classical independence assumption. To fix ideas, we assume

$$\frac{D(F_1, F_2, \dots, F_l)}{D(x_1, x_2, \dots, x_l)} \neq 0 \quad \text{in } x^0 = (x_1^0, x_2^0, \dots, x_d^0). \quad (2)$$

The hypothesis (2) will be dropped in the next section.

Clearly, condition (2) remains valid on a neighbourhood  $V \in \mathcal{V}(x^0)$ ,  $V \subset \Omega$ , under the  $C^1(\Omega)$  assumption on  $F_j(\cdot)$ ,  $j = \overline{1, l}$  and we denote by  $A(x)$ ,  $x \in V$ , the corresponding nonsingular  $l \times l$  matrix from (2).

We introduce on  $V$  the linear systems of equations with unknowns  $v(x) \in R^d$ ,  $x \in V$ :

$$v(x) \cdot \nabla F_j(x) = 0, \quad j = \overline{1, l}. \quad (3)$$

We shall use  $d - l$  solutions of (3) obtained by fixing successively the last  $d - l$  components of the vector  $v(x) \in R^d$  to be the rows of the identity matrix in  $R^{d-l}$  multiplied by  $\Delta(x) = \det A(x)$ . Then, the first  $l$  components are uniquely determined, by inverting  $A(x)$ , due to (2).

In this way, the obtained  $d - l$  solutions of (3), denoted by  $v_1(x), \dots, v_{d-l}(x) \in R^d$ , are linear independent, for any  $x \in V$ .

Moreover, these vector fields are continuous in  $V$  as  $\nabla F_j(\cdot)$  are continuous in  $V$  and the Cramer rule ensures the continuity of the solution for linear systems with respect to the coefficients. Other choices of solutions for (3), useful in this section, are possible (see Theorem 7).

We introduce now  $d - l$  nonlinear systems of first order partial differential equations associated to the vector fields  $(v_j(x))_{j=\overline{1, d-l}}$ ,  $x \in V \subset \Omega$ . Furthermore, we denote the sequence of independent variables by  $t_1, t_2, \dots, t_{d-l}$ .

These systems have a nonstandard (iterated) character in the sense that the solution of one of them is used as initial condition in the next one. Consequently, the independent variables in the "previous" systems enter as parameters in the next system. Due to their simple structure, we stress that each system (4), (5), ..., (6), may be interpreted as an ordinary differential system in  $V \subset R^d$ , with parameters, although partial differential notations are used:

$$\frac{\partial y_1(t_1)}{\partial t_1} = v_1(y_1(t_1)), \quad t_1 \in I_1 \subset R, \quad (4)$$

$$y_1(0) = x^0;$$

$$\frac{\partial y_2(t_1, t_2)}{\partial t_2} = v_2(y_2(t_1, t_2)), \quad t_2 \in I_2(t_1) \subset R, \quad (5)$$

$$y_2(t_1, 0) = y_1(t_1);$$

... ..

$$\frac{\partial y_{d-l}(t_1, t_2, \dots, t_{d-l})}{\partial t_{d-l}} = v_{d-l}(y_{d-l}(t_1, t_2, \dots, t_{d-l})), \quad (6)$$

$$t_{d-l} \in I_{d-l}(t_1, \dots, t_{d-l-1}),$$

$$y_{d-l}(t_1, \dots, t_{d-l-1}, 0) = y_{d-l-1}(t_1, t_2, \dots, t_{d-l-1}).$$

Here, the notations  $I_1, I_2(t_1), \dots, I_{d-l}(t_1, \dots, t_{d-l-1})$  are  $d-l$  real intervals, containing 0 in interior and depending, in principle, on the "previous" parameters. The existence of the solutions  $y_1, y_2, \dots, y_{d-l}$  follows by the Peano theorem due to the continuity of the vector fields  $(v_j)_{j=\overline{1, d-l}}$  on  $V$ .

The next theorems give precise details that clarify the above setting and its use.

**Theorem 1** *a) There are closed intervals  $I_j \subset R$ ,  $0 \in \text{int} I_j$ , independent of the parameters, such that  $I_j \subset I_j(t_1, t_2, \dots, t_{j-1})$ ,  $j = \overline{1, d-l}$ .*

*b) For any fixed values of the corresponding parameters, the systems (4) - (6) have solutions  $y_j \in C^1(I_j)$ ,  $j = \overline{1, d-l}$ . Moreover,  $y_j, j = \overline{1, d-l}$  are continuously differentiable in the origin with respect to all the arguments and we have:*

$$\frac{\partial y_{d-l}}{\partial t_k}(0, \dots, 0) = v_k(x^0), \quad k = \overline{1, d-l}.$$

**Proof.** Each systems (4) - (6) is solved locally in  $V$  and any point from the obtained trajectories may serve as an initial condition for the "next" system to be locally solved in  $V$  as well. The existence of local solutions is ensured by Peano theorem, which also gives an estimate of the existence intervals.

We denote by  $M = \max \left\{ |v_j|_{C(\overline{V})}, j = \overline{1, d-l} \right\}$ .

Take  $V_j, j = \overline{1, d-l-1}$  such that  $x_0 \in V_j \subset\subset V_{j+1} \subset V$ , open subsets. Let  $b_1 = \text{dist}(x^0, \partial V_1) > 0$ , then we may choose  $I_1 = \left[ -\frac{b_1}{M}, \frac{b_1}{M} \right]$  and the local solution of (4) is obtained in  $V_1$ . Fix  $b_2 = \min \{ \text{dist}(x, y); x \in \partial V_1, y \in$

$\partial V_2\}$ . Then the solution of (5) exists in  $I_2 = \left[-\frac{b_2}{M}, \frac{b_2}{M}\right]$  for any initial data from  $V_1$  and with the trajectory contained in  $V_2$ .

This argument can be iterated up to the system (6) and the number of iteration steps is finite. This proves the first point.

The regularity of the solutions (just with respect to  $t_j \in I_j$ ) is a consequence of the continuity of  $v_j(\cdot)$ ,  $j = \overline{1, d-l}$ . Clearly,  $y_1$  satisfies the last statement on differentiability in the origin. Then,  $y_2$  satisfies it as well since  $y_2(t_1, 0) = y_1(t_1)$ . And so on, this extends step by step up to  $y_{d-l}$  which is continuously differentiable in all its arguments in the origin. The last equalities follow from (4) - (6). For  $k = d-l$ , we use (6) and  $y_{d-l}(0, \dots, 0) = y_{d-l-1}(0, \dots, 0) = \dots = y_1(0) = x^0$ . For  $k = d-l-1$ , we use the initial condition and we get

$$\frac{\partial y_{d-l}}{\partial t_{d-l-1}}(0, \dots, 0) = \frac{\partial y_{d-l-1}}{\partial t_{d-l-1}}(0, \dots, 0) = v_{d-l-1}(y_{d-l-1}(0, \dots, 0)) = v_{d-l-1}(x^0).$$

This proceeds iteratively up to  $k = 1$  and ends the proof.

## Remark 2

In [19], for  $d = 3$ , two iterated Hamiltonian systems are used. A related analysis via specific ODE's arguments together with relevant numerical examples are indicated. The system (4) - (6) is a generalization of this situation and we underline that, as in [19], one can approximate easily its solution, for instance with MatLab. The system (4) is an usual ordinary differential system and we get its approximate solution in the discretization points of  $I_1$ ; then the system (5) is solved for each initial condition defined for the values of the parameter  $t_1$  given by these discretization points in  $I_1$  and so on. Finally, one obtains the approximate values of  $y_{d-l}(t_1, t_2, \dots, t_{d-l})$  on a discretization grid of  $I_1 \times I_2 \times \dots \times I_{d-l}$ . This can be achieved very simply and very quickly, by standard numerical routines for ODE's.

**Theorem 3** *For every  $k = \overline{1, l}$ ,  $j = \overline{1, d-l}$ , we have*

$$F_k(y_j(t_1, t_2, \dots, t_j)) = 0, \quad \forall (t_1, t_2, \dots, t_j) \in I_1 \times I_2 \times \dots \times I_j. \quad (7)$$

**Proof.** We notice first that, for any  $k = \overline{1, l}$ , we have:

$$\frac{\partial}{\partial t_1} F_k(y_1(t_1)) = \nabla F_k(y_1(t_1)) \cdot v_1(y_1(t_1)) = 0, \forall t_1 \in I_1,$$

since  $v_1$  is orthogonal to  $\nabla F_k$ ,  $k = \overline{1, l}$ , by (3).

Moreover,  $F_k(y_1(0)) = F_k(x^0) = 0$ ,  $k = \overline{1, l}$ , by (1.1). This gives (2.6) for  $j = 1$ . The argument follows by induction after  $j$ :

We assume that for  $j = \overline{1, r}$ ,  $r \in N$ ,  $r \leq d - l - 1$ , we have (7) for any  $k = \overline{1, l}$  and for any  $(t_1, t_2, \dots, t_r) \in I_1 \times I_2 \times \dots \times I_r$ .

We show that this is also valid for  $j = r + 1$ . First we remark that

$$F_k(y_{r+1}(t_1, t_2, \dots, t_r, 0)) = F_k(y_r(t_1, t_2, \dots, t_r)) = 0, \forall k = \overline{1, l}, \quad (8)$$

due to the induction hypothesis.

We also notice that

$$\begin{aligned} \frac{\partial}{\partial t_{r+1}} F_k(y_{r+1}(t_1, t_2, \dots, t_{r+1})) = \\ = \nabla F_k(y_{r+1}(t_1, t_2, \dots, t_{r+1})) \cdot v_{r+1}(y_{r+1}(t_1, t_2, \dots, t_{r+1})) = 0, \end{aligned} \quad (9)$$

$$\forall (t_1, t_2, \dots, t_{r+1}) \in I_1 \times I_2 \times \dots \times I_{r+1}, \forall k = \overline{1, l},$$

due to the differential equation satisfied by  $y_{r+1}$  on  $I_{r+1}$  and to the orthogonality relation (3) satisfied by the construction of  $v_{r+1}(\cdot)$ .

By (8), (9), we get (7) for  $j = r + 1$  and this ends the proof.

#### Remark 4

This theorem shows that  $F_k$ ,  $k = \overline{1, l}$ , are prime integrals for each of the  $d - l$  ordinary differential systems defined in (4) - (6). In particular, each of the systems in (4) - (6) solves the "inverse" problem: given  $F_k$ ,  $k = \overline{1, l}$  and  $x^0 \in \Omega$  such that  $F_k(x^0) = 0$ ,  $k = \overline{1, l}$ , find certain differential systems such that  $F_k$ ,  $k = \overline{1, l}$ , are prime integrals for them. The importance of the orthogonality properties (3) for prime integrals is well known, Barbu [1], Hartman [12].

#### Remark 5

Under hypothesis (2), the local solution of (1) is a  $d - l$  dimensional manifold around  $x^0$ . We expect that  $y_{d-l}(t_1, t_2, \dots, t_{d-l})$  is a local parametrization of this manifold on  $I_1 \times I_2 \times \dots \times I_{d-l}$ .

**Theorem 6** *If  $F_k \in C^2(\Omega)$ ,  $k = \overline{1, l}$ , and  $I_j$  are sufficiently small,  $j = \overline{1, d-l}$ , then the mapping*

$$y_{d-l} : I_1 \times I_2 \times \dots \times I_{d-l} \rightarrow R^d$$

*is regular and one-to-one on its image.*

**Proof.** If  $\nabla F_k \in C^1(\Omega)^d$  for all  $k = \overline{1, l}$ , then  $v_j \in C^1(V)^d$  for any  $j = \overline{1, d-l}$ , due to (2) and (3) and the way we choose the independent solutions in (3).

Then, the Cauchy-Lipschitz theorem gives the existence and the uniqueness of the solutions for all the  $d-l$  nonlinear ordinary differential systems in (4) - (6). Moreover, we also get that  $y_j \in C^1(I_1 \times I_2 \times \dots \times I_j)$  by standard differentiability properties with respect to the initial conditions applied in each subsystem, Barbu [1], Hartman [12], and an induction argument.

The matrix  $B$  of partial derivatives of  $y_{d-l} = (y_{d-l}^1, y_{d-l}^2, \dots, y_{d-l}^d)$ , where the superscripts denote the components of the vector  $y_{d-l}$ , is:

$$B = \begin{pmatrix} \frac{\partial y_{d-l}^1}{\partial t_1} & \frac{\partial y_{d-l}^1}{\partial t_2} & \dots & \frac{\partial y_{d-l}^1}{\partial t_{d-l}} \\ \dots & \dots & \dots & \dots \\ \frac{\partial y_{d-l}^d}{\partial t_1} & \frac{\partial y_{d-l}^d}{\partial t_2} & \dots & \frac{\partial y_{d-l}^d}{\partial t_{d-l}} \end{pmatrix} \quad (10)$$

The dimension of  $B$  is  $d \times (d-l)$ . We denote by  $M_{d-l}$  the  $(d-l) \times (d-l)$  matrix of the last  $d-l$  rows in  $B$  and we compute its determinant. Notice that the last column in  $M_{d-l}$  is given by the last  $d-l$  components of the vector  $v_{d-l}$ , that is  $(0, 0, \dots, 0, \Delta(x))^T$  due to the way we have constructed  $v_{d-l}$  in (3),  $x$  being here the appropriate point in  $V$  obtained as the value of the solution  $y_{d-l}(\overline{t_1}, \overline{t_2}, \dots, \overline{t_{d-l}})$ , for some  $(\overline{t_1}, \overline{t_2}, \dots, \overline{t_{d-l}}) \in I_1 \times I_2 \times \dots \times I_{d-l}$ . We write shortly  $\Delta(y_{d-l})$  for  $\Delta(x)$  with  $x$  determined as above. We cut the last row and the last column in  $M_{d-l}$ , we denote the obtained matrix by  $M_{d-l-1}$  and we have:

$$\det M_{d-l} = \Delta(y_{d-l}) \det M_{d-l-1}. \quad (11)$$

Taking into account the equation of  $y_{d-l}$  (see (6)) and the fact that the components of  $v_{d-l}$ , from order  $l+1$  to order  $d-1$  are 0 (as mentioned



above), the initial condition in (6) gives by integration:

$$y_{d-l}^{l+1} = y_{d-l-1}^{l+1}; \dots; y_{d-l}^{d-1} = y_{d-l-1}^{d-1}$$

and they are independent of  $t_{d-l}$ . Therefore, we can write

$$M_{d-l-1} = \begin{pmatrix} \frac{\partial y_{d-l-1}^{l+1}}{\partial t_1} & \frac{\partial y_{d-l-1}^{l+1}}{\partial t_2} & \dots & \frac{\partial y_{d-l-1}^{l+1}}{\partial t_{d-l-1}} \\ \dots & \dots & \dots & \dots \\ \frac{\partial y_{d-l-1}^{d-1}}{\partial t_1} & \frac{\partial y_{d-l-1}^{d-1}}{\partial t_2} & \dots & \frac{\partial y_{d-l-1}^{d-1}}{\partial t_{d-l-1}} \end{pmatrix}. \quad (12)$$

Relation (12) shows that in fact  $M_{d-l-1}$  has a similar structure as  $M_{d-l}$ , associated to  $y_{d-l-1}$ . Using the differential system satisfied by  $y_{d-l-1}$  and the structure of  $v_{d-l-1}$ , we see again that the last column in  $M_{d-l-1}$  is of the form  $(0, 0, \dots, 0, \Delta(y_{d-l-1}))^T$  (and of length  $d-l-1$ ).

One can iterate the above arguments to obtain

$$\det M_{d-l} = \Delta(y_{d-l}) \det M_{d-l-1} = \Delta(y_{d-l}) \Delta(y_{d-l-1}) \det M_{d-l-2} = \quad (13)$$

$$\dots = \Delta(y_{d-l}) \Delta(y_{d-l-1}) \dots \Delta(y_1) \neq 0,$$

where the notations  $M_{d-l-2}$ , etc., are obvious. Relations (10) - (13) end the proof.

We consider now another solution choice in (3). We shall use  $d-l$  solutions of (3) obtained by fixing the last  $d-l$  components of the vector  $v(x) \in R^d$  to be the rows of the identity matrix in  $R^{d-l}$ . All the above arguments can be adapted to this choice as well. Moreover, the next result shows that we construct exactly the solution of the classical implicit functions theorem, which follows as a special case of our approach.

**Theorem 7** *If  $F_k \in C^1(\Omega)$ , the last  $d-l$  components of  $y_{d-l}$  have the form  $(t_1 + x_{l+1}^0, t_2 + x_{l+2}^0, \dots, t_{d-l} + x_d^0)$ , that is the first  $l$  components of  $y_{d-l}$  give the unique solution of the implicit system (1) on  $(x_{l+1}^0, x_{l+2}^0, \dots, x_d^0) + (I_1 \times I_2 \times \dots \times I_{d-l})$ .*

**Proof.** By inspection and induction, one can see that the last  $d - l$  components of  $y_j(t_1, t_2, \dots, t_j)$  are  $(t_1 + x_{l+1}^0, t_2 + x_{l+2}^0, \dots, t_j + x_{l+j}^0, t_{j+1} + x_{l+j+1}^0, \dots, t_{d-l} + x_d^0)$ .

This is due to the special choice of the last components of the vectors  $v_k$  in (2.2), as rows of the identity matrix, allowing explicit integration. Then, we have just to remark that by redenoting the last  $d - l$  components of  $y_{d-l}$  as  $(s_{l+1}, s_{l+2}, \dots, s_d)$ , then the first  $l$  components of  $y_{d-l}$  are functions of  $(s_{l+1}, s_{l+2}, \dots, s_d)$ , defined on  $(x_{l+1}^0, x_{l+2}^0, \dots, x_d^0) + (I_1 \times I_2 \times \dots \times I_{d-l})$ , solving (1) due to Theorem 3.

The uniqueness comes from the implicit function theorem. It is enough to impose  $C^1$  assumptions since Theorem 6 is not used here.

#### Remark 8

In particular, we also get that the differential system (4)-(6) has unique solution in this case, although the right-hand side is just continuous. Again by the implicit functions theorem (under hypothesis (2) for  $C^1$  functions) and by the relation

$$y_{d-l}(t_1, t_2, \dots, t_j, 0, \dots, 0) = y_j(t_1, t_2, \dots, t_j)$$

we see that  $y_1, y_2, \dots, y_{d-l}$  have continuous partial derivatives with respect to their arguments. They can be computed according to the classical formulas.

#### Remark 9

In dimension two or three, in [19], [29], variants of Theorem 6 are studied that provide general parametrizations.

We underline that, although Theorem 7 provides the classical solution of the implicit functions theorem, a parametrization may be more advantageous in applications since it offers a more complete description of the corresponding manifold by removing the condition to obtain just functions. One can use maximal solutions of (4) - (6) and, in many examples, the (local) maximal solution from Theorem 6 may give even a global description of the manifold, [19], [29]. In applications, the choice of other solutions of (3) is also possible [20], in order to improve the description of the manifold.

#### Remark 10

Beside the existence statement, Theorem 7 gives a construction recipe for the implicit solution and an evaluation of the existence neighborhood (via Theorem 1), in the system (1).

For instance, if in the proof of Theorem 1 we take  $V = B(x_0, R)$  and  $V_j = B(x_0, jR(d-l)^{-1})$ , then  $I_j = [-R/(d-l)M, R/(d-l)M]$ , for  $j = 1, 2, \dots, d-l$ . This may be compared with [4], [23] where other types of arguments are used.

We consider now general perturbations of (1) having the form

$$F_k^\lambda(x_1, \dots, x_d) = 0, \quad k = \overline{1, l}, \quad \lambda \in (-1, 1), \quad (14)$$

where  $F_k^\lambda \in C^2(\Omega \times (-1, 1))$ ,  $F_k^0 = F_k$  and  $F_k^\lambda(x^0) = 0$ ,  $k = \overline{1, l}$ . Hypothesis (2) remains clearly valid for the perturbation as well, for  $\lambda$  small.

We denote by  $(S_\lambda)$  the differential system similar to (4) - (6), associated to the perturbed implicit system (14) and by  $v_j^\lambda$  the corresponding solutions of (3), appearing in the right-hand side of  $(S_\lambda)$ . Then,  $v_j^\lambda$  are in  $C^1(V_1 \times (-\lambda_0, \lambda_0))$ , under our hypotheses, for some  $V_1 \in \mathcal{V}(x^0)$ ,  $V_1 \subset\subset V$  independent of  $\lambda \in (-\lambda_0, \lambda_0)$ , for  $\lambda_0$  small. The same ideas as in Thm. 1 or Rem. 10 and the obvious property

$$M^\lambda = \max \left\{ |v_j^\lambda|_{C(\overline{V})}, j = \overline{1, d-l} \right\} \rightarrow M = \max \left\{ |v_j|_{C(\overline{V})}, j = \overline{1, d-l} \right\}$$

give the existence of the closed intervals with the origin in their interior  $I_j, j = \overline{1, d-l}$ , independent of  $\lambda$ , such that the solution of  $S_\lambda$  is defined on  $I_1 \times I_2 \times \dots \times I_{d-l}$ .

We denote by  $y_1^\lambda(t_1), \dots, y_{d-l}^\lambda(t_1, t_2, \dots, t_{d-l})$ , the unique (due to the regularity  $v_j^\lambda \in C^1(V_1 \times (-\lambda_0, \lambda_0))$ ) solution of  $(S_\lambda)$ , defined in  $I_1 \times \dots \times I_{d-l}$ . By making translations with respect to the initial conditions in each subsystem of  $(S_\lambda)$ , the initial conditions become 0 and the differentiability properties of the solution, with respect to  $\lambda$ , are a consequence of standard results on the differentiability with respect to the parameters in ODE's (again since  $v_j^\lambda \in C^1(V_1 \times (-\lambda_0, \lambda_0))$ ) and of an inductive argument as before. Denoting by  $z_1^\lambda(t_1), \dots, z_{d-l}^\lambda(t_1, t_2, \dots, t_{d-l})$  the derivative of the above solution with respect to  $\lambda \in (-\lambda_0, \lambda_0)$ , the system in variations associated to (14) and (1) can be obtained by differentiation in  $(S_\lambda)$  with respect to  $\lambda$  of the perturbations  $v_j^\lambda, j = \overline{1, d-l}$ , etc. For the case of the implicit function theorem (i.e. Theorem 7), we obtain explicit information in algebraic form :

**Proposition 11** *We have:*

- a) *the last  $d - l$  components of  $z_1^\lambda(t_1), \dots, z_{d-l}^\lambda(t_1, t_2, \dots, t_{d-l})$  are null.*
- b) *for any  $j = 1, \dots, d - l$  and  $(t_1, t_2, \dots, t_{d-l}) \in I_1 \times \dots \times I_{d-l}$ ,  $z_j^\lambda(t_1, t_2, \dots, t_j)$  is the unique solution of:*

$$\nabla_y F_k^\lambda(y_j^\lambda) z_j^\lambda + \partial_\lambda F_k^\lambda(y_j^\lambda) = 0, \quad k = 1, \dots, l. \quad (15)$$

**Proof.** The first statement is a clear consequence of Theorem 7 and of the above discussion. Since we have already established above the differentiability properties of  $y_j^\lambda$  with respect to  $\lambda$  on some given open set, then, one can differentiate with respect to  $\lambda$  in (14) with  $x_j$  replaced by  $y_j^\lambda$ , to obtain (15). Notice that the solution of the linear system (15) is unique due to (2) and to a).

**Remark 12**

One can obtain for  $z_1^\lambda(t_1), \dots, z_{d-l}^\lambda(t_1, t_2, \dots, t_{d-l})$  the relation (15) even for implicit parametrizations as in Theorem 6, but point a) is not valid and (15) is not uniquely determining (without supplementary information)  $z_1^\lambda(t_1), \dots, z_{d-l}^\lambda(t_1, t_2, \dots, t_{d-l})$ .

To obtain the necessary supplementary information, one has to use directly the differential systems (4)-(6) and to compute the corresponding system in variations.

As an example, consider now the special case of perturbations of the form

$$F_j(x_1, \dots, x_d) + \lambda h_j(x_1, \dots, x_d) = 0, \quad j = \overline{1, l}, \quad \lambda \in (-1, 1), \quad (16)$$

where  $h_j \in C^2(\Omega)$ ,  $h_j(x^0) = 0$ .

If, moreover,  $l = 1$  and the equation  $F(x_1, \dots, x_d) = 0$ ,  $F \in C^2(\Omega)$ , together with the associated initial condition, represents the boundary of a subdomain in  $\Omega$  (where  $F < 0$ , for instance) then the geometric perturbation defined by (16) may be very complex, including topological and boundary perturbations [18], [13], [26]. The perturbations (16) correspond to a directional derivative in the implicit system (1). Consequently, we may define, for  $l = 1$ , a new type of geometric directional derivative of domains, very general with respect to the admissible geometric variations.

### 3 Generalized solutions

In this section, we discuss the problem (1) for  $F_j \in C^1(\Omega)$ ,  $j = \overline{1, l}$ , in the absence of the hypothesis (2) i.e. we may have  $\det A(x^0) = 0$  (in fact all determinants of maximal order  $l$  may be null in  $x^0$ ). We remark that there is  $\{x^n\} \subset \Omega$ , such that:

$$x^n \rightarrow x^0, \quad \text{rank} J(x^n) = l, \quad n \in N, \quad (17)$$

where  $J(x^n)$  denotes the Jacobian matrix of  $F_1, F_2, \dots, F_l \in C^1(\Omega)$ , in  $x^n$ .

Notice that in case (17) is not fulfilled, it means that  $\text{rank } J(x) < l$  in  $x \in W$ , where  $W$  is a neighbourhood of  $x^0$ . Then  $F_1, F_2, \dots, F_l$  are not functionally independent in  $W$  and the problem (1) can be reformulated by using less functionals [22], [24]. That is (17) is in fact always valid, except for not well formulated problems. One may classify the systems of type (1), from this point of view, in well-posed and ill-posed systems. Notice as well that (17) is fulfilled if (2) holds, i.e. (17) is the generalization of (2), valid for all well-posed implicit systems.

Due to (17), in each  $x^n$ , one can use Theorem 7 (or the implicit functions theorem) for the system

$$F_j(x) - F_j(x^n) = 0, \quad j = \overline{1, l}, \quad x \in \overline{\Omega}, \quad (18)$$

where we can find locally the solution of (18) around  $x^n$ , in a neighborhood depending on  $n$ .

From (17), we also have  $F_j(x^n) \rightarrow F_j(x^0) = 0$ , for  $n \rightarrow \infty$ ,  $j = \overline{1, l}$ , since  $F_j \in C^1(\Omega)$ .

We denote by  $T_n$  the closure in  $\overline{\Omega}$  of the manifold defined by (18). It is compact and connected. We also have that  $\{T_n\}$  are uniformly bounded since  $\Omega$  is bounded and, on a subsequence denoted by  $\alpha$ , we get

$$T_n \rightarrow T_\alpha, \quad n \rightarrow \infty, \quad (19)$$

in the Hausdorff-Pompeiu metric [18], [15], where  $T_\alpha$  is some compact connected subset in  $R^d$ .

**Definition 13**  $T = \bigcup_{\alpha} T_\alpha$  is the local generalized solution of (1) in  $x^0$ . The union is taken for all the sequences and subsequences satisfying (17), (19).

This notion was introduced in [29] and further discussed in [19], in dimension two and three, by exploiting continuity properties with respect to data in Hamiltonian systems. The present treatment in arbitrary dimension is based on general convergence properties and allows a relaxation of the regularity conditions.

**Remark 14**

The above definition covers all critical or non critical cases. See Remark 16 as well. For instance, if in (1) we have just one equation and  $x^0$  is an isolated extremum for the respective function, then the generalized solution is just  $\{x^0\}$ . If the respective function is identically zero in the open set  $O \subset \Omega$  and  $x^0$  is on the boundary of  $O$ , then (17) is satisfied and the generalized solution is the boundary of  $O$  or some subset of it - see Proposition 15 and Example 17 below. A complete description of the level sets (even of positive Lebesgue measure) may be obtained in this way via the generalized solutions. Generally speaking, the generalized solution is not a manifold and may be not a compact subset (if  $\Omega$  is unbounded), but it is connected. The approximating generalized solution, i.e.  $\bigcup T_{n_0}$  (for some "big"  $n_0$  in (19) and for various choices of the approximating sequences of  $x_0$  in (17)), may be not connected. One can easily approximate the generalized solutions, by the techniques from section 2 applied to the corresponding terms from the sequence  $\{x_n\}$  close enough to  $x^0$ . Due to the properties of the Hausdorff-Pompeiu distance, the approximation is uniform in the space variables. If not enough sequences are taken into account, it is possible to obtain (locally) just a subset of  $T$ . For instance, in the equation  $x^2 - y^2 = 0$ , around the origin, with one approximating sequence  $(x_n, y_n) \rightarrow (0, 0)$ , such that  $|y_n| < x_n$ , just some part of the solution is generated at the limit. Taking into account a supplementary sequence such that  $x_n < -|y_n|$  the whole solution is obtained (locally) by Definition 13. An algorithm for the approximation of the generalized solution is discussed in [20], including many relevant examples.

Let  $M \subset \overline{\Omega}$  denote the connected component of the solution of (1), containing the critical point  $x^0$ . If  $\text{int}M$  is nonvoid, then it does not contain  $x^0$ , due to (17).

**Proposition 15** *We have:  $x^0 \in T_\alpha \subset T \subset \partial M_{x_0}, \forall \alpha$ , where  $\partial M_{x_0}$  is the connected component of  $\partial M$  containing  $x_0$ . In particular*

$$F_j(x) = 0, \quad j = \overline{1, l}, \quad \forall x \in T. \quad (20)$$

**Proof.** By (17), we have  $x^n \in T_n, \forall n$  and we get  $x^0 \in T_\alpha$  by the definition of the Hausdorff-Pompeiu convergence. The next inclusion follows by Definition 13.

The same argument gives that, for any  $x \in T$ , then  $x \in T_\beta$  for some subsequence  $\beta$ , and there are  $\lambda_n \in T_n$  (here  $T_n$  is the subsequence convergent to  $T_\beta$ ) such that  $\lambda_n \rightarrow x$  for  $n \rightarrow \infty$ . By (18), we see that  $F_j(\lambda_n) = F_j(x^n) \rightarrow F_j(x^0) = 0, j = \overline{1, l}$ , on a subsequence. Then, by continuity,  $F_j(\lambda_n) \rightarrow F_j(x) = 0$  as claimed and (20) is proved.

Consequently,  $T_\alpha \subset M, \forall \alpha$ . If  $\text{int}M$  is nonvoid, then it is formed just of points not satisfying (2) since  $\nabla F_j$  are null. Then  $\{x^n\}$  are disjoint from  $\overline{\text{int}M}$  ( $M$  is not necessarily a Caratheodory set and may be distinct from  $\overline{\text{int}M}$ ) and, consequently,  $T_\alpha \subset \partial M$ . By Prop.A3.2 in [18], each  $T_\alpha$  is connected and contains  $x_0$ , by the above argument. If  $\partial M$  has more connected components, then it yields  $T_\alpha \subset \partial M_{x_0}, \forall \alpha$ . Definition 13 ends the proof.

**Remark 16**

If  $x^0$  is a regular point, i.e. (2) is satisfied, then we denote by  $S$  the manifold giving the (local) solution of (1) around  $x^0$ . Then  $S$  coincides with the generalized solution around  $x^0$ .

In Definition 13, we may also choose  $x^n \rightarrow x^0, x^n \in S$  and the uniqueness property from the implicit functions theorem gives (for this choice) that  $T_n = S$  locally, for  $n$  big enough. This choice of  $\{x^n\}$  satisfies (17) since  $J(x^n) \rightarrow J(x^0)$ , so  $x^n$  satisfies (2) for  $n$  big enough. We see that in the classical case, one obtains  $T = S$  (locally), that is Definition 13 gives indeed a generalization of the classical local solution of the implicit functions theorem.

**Example 17** *In  $R^2$ , take  $d = 2, l = 1$  and*

$$f(x_1, x_2) = \begin{cases} x_1^2(x_2^2 - x_1^2)^2 & \text{if } x_1 < 0, \quad |x_2| \leq |x_1| \\ 0 & \text{otherwise.} \end{cases} \quad (21)$$

Clearly  $f$  is in  $C^1(R^2)$  and  $\nabla f(x_1, x_2) = 0$ , on the second line of (21). Take  $x^0 = (0, 0)$  and  $x^n \rightarrow x^0$ ,  $x^n = (x_1^n, x_2^n)$ ,  $x_1^n < 0$ ,  $|x_2^n| < |x_1^n|$ .

In such points  $x^n$ , one can use Theorem 6 and (3), together with the relations (4) - (6), give the Hamiltonian system (in dimension two, iterated systems are not necessary):

$$\begin{aligned} x_1'(t) &= -4x_1^2x_2(x_2^2 - x_1^2), \\ x_2'(t) &= 2x_1(x_2^2 - x_1^2)(x_2^2 - 3x_1^2), \\ (x_1(0), x_2(0)) &= x^n. \end{aligned} \quad (22)$$

Here, we have chosen  $(-f_{x_2}, f_{x_1})$  as the solution of (3).

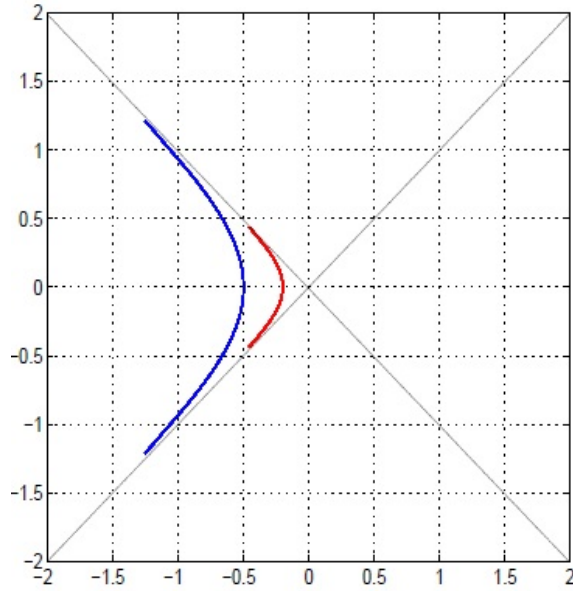


Figure 1:

In Figure 1, we represent the solution  $T_n$  of (22) obtained with MatLab, for  $x^n = (-\frac{1}{n}, 0)$ ,  $n = 2, 5$ . The generalized solution of the implicit function problem (1) corresponding to (21) is given by  $T = \{(x_1, x_2) \in R^2; x_1 = \pm x_2, x_1 \leq 0\}$ , the boundary of the critical set of  $f(\cdot, \cdot)$ , to which  $x^0$  belongs.



The generalized solution contains the essential information about the solution set of (1), since it gives its boundary (and in Proposition 15 the inclusion becomes equality, in this example).

If we define

$$f_1(x_1, x_2) = x_1^2[(x_1^2 + x_2^2 - 1)_+]^2$$

and  $x^0 = (0, 1)$ , then  $\partial M$  is connected and the corresponding generalized solution is  $\partial M$  without the lower half of the unit circle. The inclusion in Proposition 15 is strict and  $M$  is not Caratheodory, in this case. This is also related to the local character of the construction from Section 2. See Ex. 2 in [19] as w

We continue now with a partial converse of Proposition 15 that shows that the notion of generalized solution is a strict extension of the classical notion of solution.

**Proposition 18** *Let  $x^0$  be the unique critical point of (1) in the closed ball  $B(x^0)$ . Then,  $T = M$  in  $B(x^0)$ .*

**Proof.** Due to Proposition 15, we have just to prove  $M \subset T$ .

Let  $A$  be a connected component of  $M - \{x^0\}$ . It is open in the relative topology of  $M \cap B(x^0)$  since all the points except  $x^0$  are regular and the implicit functions theorem can be applied. It is also maximal in the sense that it cannot be strictly extended in  $B(x^0)$ . Notice that in the relative topology of  $M$ , we have  $\partial A \subset \partial B(x^0) \cup \{x^0\}$ , by the implicit function theorem. Consequently,  $\overline{A} \subset A \cup \{x^0\}$  since  $A$  is maximal and the part of  $\partial A$  contained in  $\partial B(x^0)$  is also contained in  $A$ .

We have  $x^0 \in \overline{A}$ . Otherwise, by the above relation, it yields  $A = \overline{A}$ , that is  $A$  is both closed and open in  $M$  and this contradicts  $M$  connected.

One can consider a sequence  $x^n \in A, x^n \rightarrow x^0$  and the associated manifolds  $T_n$ . Notice that  $A = T_n$  by the implicit functions theorem. It follows that  $A \cup \{x^0\} = \lim \overline{T_n}, A \cup \{x^0\} \subset T$ . As  $A$  is an arbitrary component of  $M - \{x_0\}$ , we get the conclusion and finish the proof.

**Proposition 19** *Let  $F_j \in C^1(\Omega)$ ,  $j = \overline{1, l}$  and  $x^n \rightarrow x^0, x^n, x^0 \in \Omega$ . Denote by  $\tilde{T}_n, \tilde{T}_0$  the generalized solutions of (1) contained in the bounded domain  $\Omega$ , corresponding to the initial conditions  $x^n$ , respectively  $x^0$ . Then*

$$\limsup_{n \rightarrow \infty} \tilde{T}_n \subset \tilde{T}_0. \quad (23)$$

**Proof.** Let  $\hat{x}_{n_k} \in \tilde{T}_{n_k}$ ,  $\hat{x}_{n_k} \rightarrow \hat{x}$ , where  $n_k \rightarrow \infty$  is some subsequence. We show that  $\hat{x} \in \tilde{T}_0$ .

By Definition 13, there is  $\tilde{x}_{n_k} \in \Omega$ , such that (2) is satisfied in  $\tilde{x}_{n_k}$  and  $|\tilde{x}_{n_k} - x^{n_k}| < \frac{1}{n_k}$  (here, we also use the characterization of the Hausdorff-Pompeiu limit) and there are  $y_{n_k} \in T_{\tilde{x}_{n_k}}$  such that  $|y_{n_k} - \hat{x}_{n_k}| < \frac{1}{n_k}$ . Consequently,  $y_{n_k} \rightarrow \hat{x}$  for  $n_k \rightarrow \infty$ . Here,  $T_{\tilde{x}_{n_k}}$  is the solution of (18) corresponding to  $\tilde{x}_{n_k}$ . By using the sequences  $\tilde{x}_{n_k} \rightarrow x^0$  and  $y_{n_k} \in T_{\tilde{x}_{n_k}}$ ,  $y_{n_k} \rightarrow \hat{x}$ , we see that  $\hat{x} \in T_0$  due to Definition 17 and the proof is finished.

**Example 20** Let  $x_n \rightarrow 0, x_n < 0$  be some strictly increasing sequence and  $g_n : R^2 \rightarrow R$  be given by

$$g_n(x, y) = \begin{cases} c_n[(x - x_n)^2 + y^2 - \frac{1}{4} \min\{|x_{n+1} - x_n|^2; |x_n - x_{n-1}|^2\}]^2, & \text{if} \\ |x - x_n|^2 + |y|^2 \leq \frac{1}{4} \min\{|x_{n+1} - x_n|^2; |x_n - x_{n-1}|^2\} \\ 0 & \text{otherwise,} \end{cases}$$

where  $c_n > 0$  is some "big" constant. We consider the function  $F : R^2 \rightarrow R$  by

$$F(x, y) = f(x, y) + \sum_{n=1}^{\infty} g_n(x, y), \quad (24)$$

where  $f$  is given in (21). Clearly  $F$  is in  $C^1(R^2)$  and  $(x_n, 0)$  are local maximum points of  $F$  if  $c_n$  are "big". The sum in (24) has always just maximum two non zero terms due to the form of the supp  $g_n$ .

Take the sequence  $x^n = (x_n, 0) \rightarrow (0, 0) = x^0$ . Then, the implicit equations  $F(x, y) = F(x_n, 0)$  have the unique solution  $x^n = (x_n, 0)$  in a neighbourhood of  $x^n$  and  $T_{x^n} = (x_n, 0)$ . In the point  $(0, 0)$ , we have  $T_0$  as in Example 17. We see in this example that the inclusion in (23) may be strict.

## 4 Reduced gradients in nonlinear programming

In constrained optimization, projected gradient methods are a classical tool, but their application may be hindered by the difficulty to effectively compute projections on the admissible set, Ciarlet [5]. Based on the results from the

previous sections, we indicate here an approach to eliminate, totally or partially, the constraints, that allows optimality conditions and sensitivity results in a more effective way, decreasing the dimension. An algorithm and relevant numerical examples are also discussed.

We study first just the case of equality and/or inequality constraints (no abstract constraints) with the aim to exploit this explicit structure. For the algorithm, our setting is general and the regularity conditions are weak. In the recent papers [27], [17], dimensional reduction is obtained via new relaxation procedures associated to implicit functions and our approach is certainly different.

We consider now the classical minimization problem with equality constraints:

$$(P) \quad \text{Min}\{g(x_1, \dots, x_d)\}$$

subject to (1). Notice that hypothesis (2) is exactly the form of the classical Mangasarian-Fromowitz condition for the special case of problem (P). By Theorem 7 we can replace it by the unconstrained problem for  $(t_1, t_2, \dots, t_{d-l}) \in (I_1 \times I_2 \times \dots \times I_{d-l})$ :

$$(P_1) \quad \text{Min}\{g(y_{d-l}^1, y_{d-l}^2, \dots, y_{d-l}^l, t_1 + x_{l+1}^0, t_2 + x_{l+2}^0, \dots, t_{d-l} + x_d^0)\},$$

where  $(y_{d-l}^1, y_{d-l}^2, \dots, y_{d-l}^l, t_1 + x_{l+1}^0, t_2 + x_{l+2}^0, \dots, t_{d-l} + x_d^0)$  are the components of  $y_{d-l}$ , the solution of (4)-(6), corresponding to this case. This is done around the point  $x^0$  that satisfies (2), which is mapped into  $(0, \dots, 0)$  via the definition of  $(P_1)$ .

**Proposition 21** *Let  $g$  and  $F_i, i = \overline{1, l}$ , be in  $C^1(R^d)$  and (2) be valid in some point  $x^0$ . Then, the gradient in  $(0, \dots, 0)$  of the composed cost functional in  $(P_1)$  has the components:*

$$\sum_{i=1}^l \frac{\partial g(x^0)}{\partial i} v_j^i(x^0) + \frac{\partial g(x^0)}{\partial(j+l)}, \quad j = \overline{1, d-l},$$

where  $\partial i$  denotes the derivative with respect to the  $i$ -th component and  $v_j$  are the vector solutions of (3) as in Thm. 7.

**Proof.** Let  $G$  denote the above composed mapping defined in a neighborhood of  $(0, 0, \dots, 0)$ . We have:

$$\frac{\partial}{\partial t_j} G(0, 0, \dots, 0) = \sum_{i=1}^l \frac{\partial g(x^0)}{\partial i} \frac{\partial y_{d-l}^i}{\partial t_j}(0, 0, \dots, 0) + \frac{\partial g(x^0)}{\partial(l+j)}(x^0).$$

Here, we have used that  $y_{d-l}(0, 0, \dots, 0) = \dots = y_1(0) = x^0$ . Applying the last conclusion in Theorem 1, we end the proof.

**Remark 22**

This methodology can be extended to the case of implicit parametrizations. The form of the above partial derivatives is due to the choice of the last  $d - l$  components of the solutions in (3) from Theorem 7 and we can write shortly

$$\frac{\partial}{\partial t_j} G(0, 0, \dots, 0) = \nabla g(x^0) \cdot v_j(x^0) \quad j = \overline{1, d-l}. \quad (25)$$

Notice that for (25), it is sufficient to solve just the  $l \times d$  linear algebraic system (3) and not the differential system (4) - (6). That is, it is very simple to implement a reduced unconstrained gradient method for the problem (P). As a byproduct, we obtain the first order optimality conditions in the Fermat form:

**Corollary 23** *If  $x^0$  is a local solution of (P) satisfying that  $g$  and  $F_i, i = \overline{1, l}$ , are in  $C^1(R^d)$  and (2) holds, then we have:*

$$\nabla g(x^0) \cdot v_j(x^0) = 0 \quad j = \overline{1, d-l}. \quad (26)$$

In fact this is equivalent with the classical Lagrange multipliers rule, since (3) produces a basis in the tangent space at  $x^0$  to the manifold defined by (1). Then (26) shows that  $\nabla g(x^0)$  is in the normal space, which has the basis given by  $\nabla F_i(x^0), i = \overline{1, l}$ , due to hypothesis (2). Consequently, there are scalars  $\lambda_i$  such that the well known relation  $\nabla g(x^0) = \sum_{i=1}^l \lambda_i \nabla F_i(x^0)$  holds. The argument works in the converse sense too. We also underline the explicit character of (26) in comparison with the Lagrange rule and its applicability to numerical minimization routines of gradient type. In view of the definition in [8], p.362, the vector appearing in the right-hand side of (25) is called the tangential gradient of  $g$  in  $x^0$ , to the manifold defined by (1) under condition (2).

We indicate now a sensitivity result for the problem (P). We consider the perturbation of (1) as defined in (14) and its assumptions. We impose the condition that

$$\nabla(\nabla g(x) \cdot v_k(x)), \quad k = \overline{1, d-l}, \quad x = x^0$$

is a basis in the tangent space in  $x^0$  to the manifold defined by (1). This hypothesis is easy to be checked (just the independence) and replaces the usual growth assumption on the Lagrangian function, see [2], § 4 since we have no multipliers here.

**Proposition 24** *For any  $\lambda > 0$  small, the perturbed optimality system*

$$\nabla g.v_k^\lambda = 0, \quad k = \overline{1, d-l}; \quad F_i^\lambda = 0, \quad i = \overline{1, l}, \quad (27)$$

(where  $v_k^\lambda$  satisfies the system (3) associated to  $F_i^\lambda, i = \overline{1, l}$ ), has a unique solution  $x^\lambda$ , differentiable with respect to  $\lambda$ . If  $w^\lambda = \frac{d}{d\lambda}x^\lambda$ , then it satisfies (for  $\lambda = 0$ ) the system

$$\nabla^2 g(x^0)w^0.v_k(x^0) + \nabla g(x^0).\nabla v_k(x^0)w^0 = 0, \quad k = \overline{1, d-l},$$

$$\nabla F_i(x^0).w^0 = 0, \quad i = \overline{1, l}.$$

**Proof.** We know by (2) that  $\nabla F_i(x_0), i = \overline{1, l}$  is a basis in the normal space to the manifold (1). Together with the above assumption on the tangent space, we get that the vectors  $\nabla F_i(x), i = \overline{1, l}$  and  $\nabla(\nabla g(x).v_k(x)), k = \overline{1, d-l}, x = x^0$  are independent in  $R^d$ . Consequently, one can apply the implicit function theorem for (27) around  $\lambda = 0, x = x^0$  and get the first conclusion. The differentiability follows again by the implicit function theorem and we end the proof by simply taking the derivative in  $\lambda = 0$ , in (27).

We discuss now the general case of both equality and inequality constraints:

$$(Q) \quad \text{Min}\{g(x_1, \dots, x_d)\}$$

subject to (1) and to

$$G_j(x) \leq 0 \quad j = \overline{1, m}, \quad (28)$$

where  $g, F_i, G_j$  are in  $C^1(R^d)$ . The Mangasarian-Fromowitz condition in this case consists of (2) and there is  $d \in R^d$  such that

$$\nabla F_i(x^0)d = 0, \quad i = \overline{1, l}, \quad \nabla G_j(x^0)d < 0, \quad j \in I(x^0), \quad (29)$$

with  $I(x^0)$  being the set of indices of active inequality constraints in  $x^0$ . See [2], § 2.3.4 or [6], § 6 for excellent presentations.

The reduced problem is again obtained via Theorem 7:

$$(Q_1) \text{ Min}\{g(y_{d-l}^1, y_{d-l}^2, \dots, y_{d-l}^l, t_1 + x_{l+1}^0, t_2 + x_{l+2}^0, \dots, t_{d-l} + x_d^0)\},$$

subject to the constraints (28), in the "reduced" form:

$$G_j(y_{d-l}^1, y_{d-l}^2, \dots, y_{d-l}^l, t_1 + x_{l+1}^0, \dots, t_{d-l} + x_d^0) \leq 0 \quad j = \overline{1, m}, \quad (30)$$

**Lemma 25** *The minimization problem  $(Q_1)$  satisfies the Mangasarian-Fromowitz condition in the origin of  $R^{d-l}$ .*

**Proof.** By the first part in (29), we see that  $d$  is in the tangent space to the manifold (1) since  $\nabla F_i(x^0), i = \overline{1, l}$  is a basis in the normal space to the manifold given (1), under hypothesis (2). Then  $d = \sum_{s=1}^{d-l} \alpha_s v_s$  with  $\alpha_s$  some scalars, since  $v_s, s = \overline{1, d-l}$ , gives a base in the tangent space.

By the second part in (29) we get  $\sum_{s=1}^{d-l} \alpha_s \nabla G_j(x^0) v_s < 0$ . Using the derivation

formula (25), this may be rewritten as  $\sum_{s=1}^{d-l} \alpha_s \frac{\partial}{\partial t_s} g_j(0, 0, \dots, 0) < 0$ , where

$$g_j(t_1, \dots, t_{d-l}) = G_j(y_{d-l}^1, y_{d-l}^2, \dots, y_{d-l}^l, t_1 + x_{l+1}^0, \dots, t_{d-l} + x_d^0).$$

is the composed mapping. This shows that the Mangasarian-Fromowitz hypothesis is satisfied in the origin of  $R^{d-l}$  with the vector  $(\alpha_1, \dots, \alpha_{d-l})$ .

If  $x^0$  is a local solution of  $(Q)$ , by Lemma 25, one can apply the classical KKT theorem to the problem  $(Q_1)$  in the origin of  $R^{d-l}$  that is a local solution. Using again the derivation formula (25), we get:

**Theorem 26** *Let  $x^0$  be a local minimum for  $(Q)$ . Then, there are  $\beta_j \geq 0, j = \overline{1, m}$  such that*

$$\begin{aligned} 0 &= \nabla g(x^0) \cdot v_s(x^0) + \sum_{j=1}^m \beta_j \nabla G_j(x^0), s = \overline{1, d-l}, \\ 0 &= \beta_j G_j(x^0), j = \overline{1, m}. \end{aligned}$$

**Remark 27**

This is a simplified version of the KKT conditions since it eliminates the Lagrange multipliers for the equality constraints. Notice that the statement is in terms of data, i.e. the problem  $(Q)$  and the differential system (4)

- (6) is not explicitly used in Theorem 26. The reduced problem is  $d - l$  dimensional. The sensitivity with respect to perturbations can be performed as in the case of pure equality constraints, as noticed in [2], p.369.

We relax now the hypotheses in the problem (Q) and we describe a direct minimization algorithm. It looks for the solution in a maximal neighborhood of  $x^0$ , where, for instance, the manifold defined by (1) avoids (partial) critical points since the maximal solutions of subsystems in (4) - (6) stop in their own (partial) critical points, giving the null value in their right-hand side. In many situations, the solution is global, see Remark 9 and [19], [29].

We assume in the sequel that  $g$  and  $G_j, j = \overline{1, m}$ , are just in  $C(R^d)$  and  $F_i, i = \overline{1, l}$ , are in  $C^1(R^d)$  and satisfy condition (2) in  $x^0$ . This last condition can be removed in fact, according to Remark 31. Notice that  $x^0$  is here just an admissible point for (Q) and not a local minimum as in Theorem 26. We can also add the abstract constraint  $x \in D$ , some given subset in  $R^d$ , such that  $x^0 \in D$ .

The main observation is that in solving numerically (4) - (6), the variant corresponding to Theorem 6, we obtain automatically a discretization of the manifold defined by (1), in a maximal neighborhood of  $x^0$ , as explained above. Let us denote by  $n$  the discretization parameter. For instance,  $1/n$  can characterize the discretization for the parameters  $t_1, \dots, t_{d-l}$ ,  $n$  or may be also linked to the length of the intervals where the maximal solution is computed, etc. While small approximation errors may appear in such computations, they don't influence essentially the optimization process due to the continuity assumption on all the involved functions in (Q). We denote by  $C_n$  the set of all these discretized points that satisfy all the constraints. They give the approximating admissible set and we formulate the algorithm:

**Algorithm 28** 1) choose  $n = 1$ , the discretization step  $1/n$  and the solution intervals  $I_1^n, \dots, I_{d-l}^n$ , the tolerance  $\delta$ , etc.  
2) compute the discrete set of admissible points  $C_n$ , starting from  $x^0$ , via (4) - (6) and (28)  
3) find in  $C_n$  the approximating minimum of (Q), denoted by  $x^n$   
4) test if the solution is satisfactory by  $|g(x_n) - g(x_{n-1})| \leq \delta$ .  
5) If YES, then STOP. If NO, then  $n := n + 1$  and GO TO step 1).

In step 4) other tests (on the solutions, on the gradients, etc.) may be used. The approximating solution  $x^n \in C_n$  may be not unique and the

Algorithm 28 finds all the solutions. One can adapt the convergence test to such situations.

**Theorem 29** *The algorithm is convergent as  $n \rightarrow \infty$ .*

This is a consequence of the density of  $\bigcup C_n$  in the admissible set, according to Theorem 6.

**Remark 30**

The set defined by the equality constraints may have several connected components. See Example 33. Starting from  $x^0$ , Algorithm 28 will minimize just on the component that contains  $x^0$ . Initial guesses from all the admissible components are necessary if we want to minimize on all of them.

**Remark 31**

If condition (2) is not fulfilled, then one can use the generalized solution of (1) as explained in Section 3, since the Hausdorff-Pompeiu distance ensures the uniform convergence of approximating points. The obtained minimum may satisfy (1) and the minimum property with some small error tolerance. An algorithm for the computation of the generalized solution, with relevant examples is studied in [20].

Finally, we comment some illustrative numerical examples.

**Example 32**

We consider first a minimization problem on the torus in  $R^3$ , with radiuses 2 and 1, defined implicitly by  $F = 0$ , and with initial point  $(x_0, y_0, z_0) = (\sqrt{5}, 2, 0)$ :

$$\begin{aligned} \min\{xyz\} \\ F(x, y, z) = (x^2 + y^2 + z^2 + 3)^2 - 16(x^2 + y^2) \end{aligned}$$

The obtained results are given below, compared with the application of the fmincon routine of MatLab:

$$\begin{aligned} \min &= -2,7154 \\ x_{\min} &= 1,7841; y_{\min} = 1,8199; z_{\min} = -0,8363 \\ f_{\mincon} : \min &= -2,7153; x_{\min} = 1,802; y_{\min} = 1,802; z_{\min} = -0.836 \end{aligned}$$



Using other starting points like  $(1, 0, 0)$  or  $(3, 0, 0)$  is not allowed by MatLab that finds no admissible solutions in these cases, while our approach works.

### Example 33

Now, we consider two equality restrictions, given by  $F$  and  $P$ , that represent a torus intersected with a paraboloid, see Fig.2 and Fig.3. Two initial points are taken into account since the intersection has two components.

$$\begin{aligned} & \min\{x^3 + 5y - 7\sin z\} \\ & P(x, y, z) = \frac{2\sqrt{3}}{3}x - y^2 - z^2 \\ & (x_0, y_0, z_0) = (\sqrt{3}, 1, 1); (x_0, y_0, z_0) = (\sqrt{3}, -1, 1) \end{aligned}$$

The numerical results and a comparison with MatLab routine `fmincon` is indicated below:

$$\begin{aligned} & (\sqrt{3}, -1, 1) : \text{minimal value} = 0.498975897823261 \\ & \text{solution} : (1.06688905550184, -0.814925789648031, 0.753631933331335) \\ & (\sqrt{3}, 1, 1) : \text{minimal value} = -7.65929313197537 \\ & \text{solution} : (1.10697710321061, 0.817093948780941, 0.781479124977557) \end{aligned}$$

In the second case `fmincon` stops after 42 iterations with the message that constraints are not satisfied within the tolerance. In the first case, `fmincon` finds basically the same solution.

### Remark 34

In [27], an example in  $R^6$ , with three equality constraints, is discussed. Reworking it via Algorithm 28, starting from the two points indicated there on p.451, we obtain the new points

$$\begin{aligned} & (0.5631, -3.2581, 0.51593, 0.4692, 1.4635, 3.589), \\ & (0.56166, -3.3154, 0.50897, 0.5047, 1.4365, 3.6777) \end{aligned}$$

with the cost values 343,7695, respectively 383,7265. This improves the quoted experiment and can be directly checked. It does not contradict [27] since our algorithm needs no bounds on the independent variables and extends the search domain, which is an advantage from the point of view

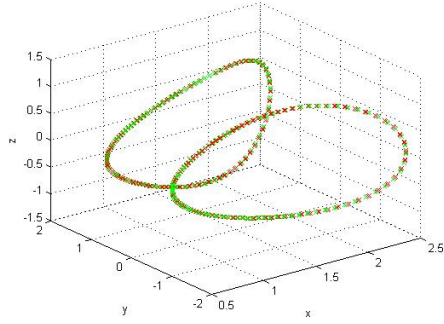


Figure 2: The admissible set

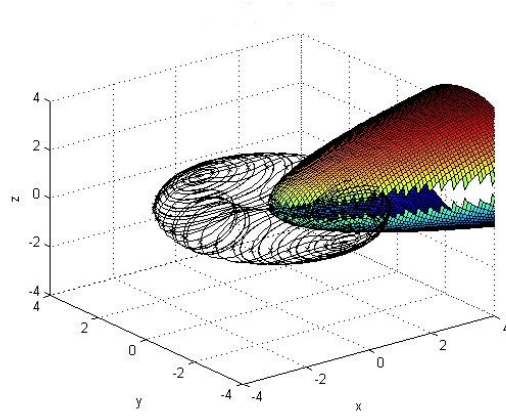


Figure 3: The geometry

of global optimization. The necessary working time, on a medium performance laptop, is several minutes. More details and some high dimensional numerical examples are indicated in [21].

## References

- [1] V. Barbu, Ecuatii diferențiale, Ed. Junimea, Iași (1985).
- [2] J.F. Bonnans and A. Shapiro, Perturbation analysis of optimization problems, Springer Verlag, New York (2000).
- [3] G.J. Butler and H.I. Freedman, Further critical cases of the scalar implicit function theorem, Aequationes Math. 8 (1972) 203211.
- [4] H.C. Chang, W. He and N. Prabhu, The analytic domain in the implicit function theorem, JIPAM, Vol. 4, Iss. 1, Article 12 (2003).
- [5] Ph. Ciarlet Introduction to numerical linear algebra and optimization, Cambridge Univ. Press, New York (1989).
- [6] F.H. Clarke Optimization and nonsmooth analysis John Wiley & Sons, New York (1983).
- [7] E. Coddington and N. Levinson, Theory of ordinary differential equations, McGraw-Hill, New York (1955).
- [8] M. Delfour and J.-P. Zolesio, Shapes and Geometry, SIAM, Philadelphia (2001).
- [9] K. Dobiasova, Parametrizing implicit curves, WDS'08 Proceedings of Contributed Papers, MATHFYZPRESS, Prague (2008), pp. 19-22.
- [10] A.L. Dontchev and R.T. Rockafellar, Implicit functions and solution mappings, Springer, New York (2009).
- [11] Xiao-Shan Gao, Conversion between implicit and parametric representations of algebraic varieties, Mathematical mechanization and applications, Academic Press, San Diego (2000), pp. 253-271.
- [12] P. Hartman, Ordinary differential equations, J. Wiley and Sons, New York (1964).
- [13] A. Henrot, M. Pierre, *Variation et optimization de formes: une analyse geometrique*, Springer Verlag, Berlin (2005).

- [14] S.G. Krantz and H.R. Parks, The implicit function theorem, Birkhäuser, Boston (2002).
- [15] C. Kuratowski, Introduction to set theory and topology, Pergamon Press, Oxford (1962).
- [16] S. Lefschetz, Differential equations: geometric theory, Interscience, New York (1957).
- [17] A. Mitsos, B. Chachuat, P.I. Barton, McCormick-based relaxations of algorithms, SIAM J. Optim. 20(2), pp.573-601 (2009).
- [18] P. Neittaanmäki, J. Sprekels, D. Tiba, Optimization of elliptic systems. Theory and applications, Springer, New York (2006).
- [19] M.R. Nicolai and D. Tiba, Implicit functions and parametrizations in dimension three: generalized solutions, DCDS - A vol. 35, no.6 (2015), pp.2701 - 2710. doi:10.3934/dcds.2015.35.2701
- [20] M.R. Nicolai, An algorithm for solving implicit systems in the critical case, Ann. Acad. Rom. Sci. Ser. Math. Appl. Vol. 7, No. 2 (2015), pp.310 - 322.
- [21] M.R. Nicolai, High dimensional applications of implicit parametrizations in nonlinear programming, to appear in "Mathematics and its Applications", vol.8, no.1 (2016).
- [22] M. Nicolescu, N. Dinculeanu and S. Marcus, Analiză Matematică, vol. I, Ed. 4, Ed. Didactică și Pedagogică, București (1971).
- [23] Phan Phien, Some quantitative results on Lipschitz inverse and implicit functions theorems, arXiv: 1204.4916v2 (2012).
- [24] W. Rudin, Principles of mathematical analysis, Second Edition, McGraw-Hill, New York (1964).
- [25] J. Schicho, Rational parametrizations of algebraic surfaces, Thesis, J. Kepler Univ. Linz (1995).
- [26] J. Sokolowski, J.-P. Zolesio, *Introduction to shape optimization. Shape sensitivity analysis*, Springer Verlag, Berlin (1992).

- [27] M.D. Stuber, J.K. Scott, P.I. Barton, Convex and concave relaxations of implicit functions, *Optimization methods and software*, 30(3), pp.424-460 (2015).
- [28] J.A. Thorpe, Elementary topics in differential geometry, Springer Verlag, New York (1979).
- [29] D. Tiba, The implicit functions theorem and implicit parametrizations, *Ann. Acad. Rom. Sci. Ser. Math. Appl.* 5, no. 1-2 (2013), pp. 193-208: <http://www.mathematics-and-its-applications.com>
- [30] D. Wang, Irreducible decomposition of algebraic varieties via characteristic set method and Gröbner basis method, *CAGD* 9 (1992), pp. 471-484.
- [31] H. Yang, B. Jüttler, L. Gonzales-Vega, An evolution-based approach for approximate parametrization of implicitly defined curves by polynomial parametric spline curves, *Math. Comp. Sci.* 4, no. 4 (2010), pp. 463-479.